10 Basic results from Linear Algebra

Theorem 10.1. *Let $A \in M^{3\times 3}$ be symmetric (i.e. $A^T = A$). Then, we have:*

1) *All eigenvalues of A are real, we define these as $\lambda_1, \lambda_2, \lambda_3$ (possibly repeated)*
2) *There is an orthonormal basis for $\mathbb{R}^3$ consisting of corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (i.e. $A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$).*

Proposition 10.2. *If $A \in M^{3\times 3}$ is symmetric with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then $\exists Q \in SO(3)$ such that $Q^T A Q = diag(\lambda_1, \lambda_2, \lambda_3) =: D$.*

Sketch proof: Define $Q := [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Then $AQ = [A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3] = [\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \lambda_3\mathbf{v}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = D$.

Now $Q^T Q = I$ and so $Q^T A Q = D$. Without loss of generality, $Q \in SO(3)$ (if $\det Q = -1$, then replace $\mathbf{v}_1$ with $-\mathbf{v}_1$)

Theorem 10.3 (Spectral Decomposition Theorem)**.** *Let $A \in M^{3\times 3}$ be symmetric with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be an orthonormal basis for $\mathbb{R}^3$ of corresponding eigenvectors. Then $A = \lambda_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 \otimes \mathbf{v}_3$. (Note: $(\mathbf{c} \otimes \mathbf{d})(\mathbf{x}) = \langle \mathbf{d}, \mathbf{x} \rangle \mathbf{c}$.)*

Proof. If $\mathbf{x} \in \mathbb{R}^3$, then $\mathbf{x} = \sum_{i=1}^3 \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$. Hence

$$A\mathbf{x} = \sum_{i=1}^{3} \langle \mathbf{x}, \mathbf{v}_i \rangle A\mathbf{v}_i = \sum_{i=1}^{3} \lambda_i \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i = \sum_{i=1}^{3} \lambda_i (\mathbf{v}_i \otimes \mathbf{v}_i)(\mathbf{x}) \tag{10.1}$$

$$= (\sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i)(\mathbf{x}) \tag{10.2}$$

Since $\mathbf{x}$ was arbitrary, the result follows. $\square$

Corollary 10.4. *For $k \in \mathbb{N}, A^k = \sum_{i=1}^3 \lambda_i^k \mathbf{v}_i \otimes \mathbf{v}_i$. This representation extends to $k \in \mathbb{Z}$ provided $\lambda_i \neq 0, \quad \forall i = 1, 2, 3$.*

Theorem 10.5 (Square root theorem)**.** *Let C be a symmetric, positive definite $n \times n$ matrix. Then there exists a unique positive definite symmetric matrix U such that $C = U^2$. (We write $C^{\frac{1}{2}} = U$.)*

Proof. Let

$$C = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$$

be the spectral decomposition of C. Notice that since C is positive-definite, all its eigenvalues are positive. Now define

$$U = \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \mathbf{v}_i \otimes \mathbf{v}_i,$$

then U is symmetric and positive-definite and satisfies $U^2 = C$. $\square$

Theorem 10.6 (Polar decomposition theorem)**.** *Let F satisfy $\det F > 0$. Then there exists $R \in SO(n)$ and positive definite symmetric matrices U, V such that $F = RU = VR$.*

Proof. Let $C = F^T F$, then C is symmetric since $C^T = (F^T F)^T = F^T F = C$ and so by the square root theorem, there exists a unique symmetric positive-definite square root U (so that $C = U^2$).

Now define $R = FU^{-1}$, then

$$R^T R = (FU^{-1})^T (FU^{-1}) = U^{-T} F^T F U^{-1} = U^{-1} C U^{-1} = I$$

So R is orthogonal and $R \in SO(n)$. By construction, we see that this decomposition is unique.

A similar proof working with $C = F F^T$ yields the (unique) decomposition $F = V\tilde{R}$ with V positive-definite and symmetric and $\tilde{R} \in SO(n)$. To see that $\tilde{R} = R$, notice that $F = \tilde{R}\tilde{R}^T V \tilde{R}$, where $\tilde{R}^T V \tilde{R}$ is positive-definite and symmetric and so by the uniqueness of the decomposition $C = RU$ it follows that $R = \tilde{R}$ and $U = \tilde{R}^T V \tilde{R}$. □